ROCA. Revista científico - educacional de la provincia Granma.
Vol.13 No.4, octubre-diciembre 2017. ISSN: 2074-0735. RNPS: 2090. roca@udg.co.cu

Revisión

"*Q*-ANALOGUE OF THE APÉRY'S CONSTANT"

"*Q*-análogos de la constante Apéry's"

Lic. Alicia María Centurión-Fajardo. Universidad de Granma. acenturionf@udg.co.cu

M. Sc. Nancy Céspedes-Trujillo. Universidad de Las Tunas, nancyct@ult.edu.cu

Lic. Eduardo Moreno-Roque. Universidad de Granma. emorenor@udg.co.cu

Recibido: 1/11/2017 Aceptado:1 6/12/201

ABSTRACT

In this paper, we give a summary introduction to the Riemann zeta function. We also provide a brief overview of the q-calculus topics which are necessary to understand the main results. Finally, we give some q-representations for the q-analogue of the Apéry's constant.

KEY WORD: Apéry's constant, Riemann zeta function, *q*-hypergeometric function.

RESUMEN

En este artículo damos un resumen introductorio de la función Zeta de Riemann. También proporcionamos una breve visión de la temática q- cálculos la cual es necesaria para un entendimiento de los principales resultados. Finalmente, damos algunas representaciones para los q-análogos de la constante Apéry´s.

PALABRAS CLAVE: constante de Apéry, función zeta de Riemann, función q-hipergeométrica.

INTRODUCTION

The Riemann zeta function [1, 2, 3, 8, 11, 14, 15] for Re $s > 1$ is defined by the series

$$\zeta\;\; s \;\equiv \underset{n\geq 1}{}\frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$$

which can be expressed as

$$\begin{aligned}\zeta(s) &= \sum_{n\geq 0}\frac{1}{(n+1)^s} = \sum_{n\geq 0}\frac{1^s2^s3^s\cdots(1+n-1)^s}{2^s3^s\cdots(2+n-2)(2+n-1)^s}\\ &= \sum_{n\geq 0}\frac{(1)_n^{s+1}}{(2)_n^s}\frac{1^n}{n!} = {}_{s+1}F_s\left(\left.\begin{array}{c}1,\ldots,1\\ \\ 2,\ldots,2\end{array}\right|1\right),\end{aligned}$$

Where $(\cdot)_k$ denotes the Pochhammer symbol, also called the shifted factorial, defined by

$$(z)_k \equiv \prod_{0\le j\le k-1} (z+j)\,, \quad k \ge 1,$$
$$(z)_0 = 1, \quad (-z)_k = 0, \quad \text{if } z < k,$$

which in terms of the gamma function is given by

$$(z)_k = \frac{\Gamma(z+k)}{\Gamma(z)}, \quad k = 0,1,2,\ldots,$$

and ${}_rF_s$ denotes the ordinary hypergeometric series [7, 10, 12] with variable z is defined by

$${}_rF_s\left(\begin{array}{c} a_1,\ldots,a_r \\ \\ b_1,\ldots,b_s \end{array}\middle|\, z\right) \equiv \sum_{k\ge 0} \frac{(a_1,\ldots,a_r)_k}{(b_1,\ldots,b_s)_k}\frac{z^k}{k!}, \qquad (1)$$

being

$$(a_1,\ldots,a_r;q)_k \equiv \prod_{1\le j\le r} (a_j;q)_k\,.$$

with $a_i\ {}_{i=1}^{r}$ and $b_j\ {}_{j=1}^{s}$ complex numbers subject to the condition that $b_j \neq -n$, with $n \in N \setminus \{0\}$ for $j = 1,2,\ldots,s$.

In particular, the Apéry's constant can be rewritten as

$$\zeta(3) = {}_4F_3\left(\begin{array}{c} 1,1,1,1 \\ \\ 2,2,2 \end{array}\middle|\, 1\right) = \sum_{k>n\ge 1} \frac{1}{k^2 n} = -\int_0^1\int_0^1 \frac{\ln x}{1-xy}dxdy. \qquad (2)$$

The structure of the paper is as follows. In Section 2, we compress some necessary definitions and tools. Finally, in Section 3, we give the main results.

1 *q*-Calculus

There is no general rigorous definition of q-analogues. An intuitive definition of a q-analogues of a mathematical object $\mathcal{G}$ is a family of objects $\mathcal{G}_q$ with 0 < q < 1, such that

$$\lim_{q\to 1^-} \mathcal{G}_q = \mathcal{G}$$

Thus, the *q*-Calculus, i.e. the *q*-analogues of the usual calculus.

Let the *q*-analogues of Pochhammer symbol [7, 10] or *q*-shifted factorial be defined by

$$(a;q)_n \equiv \begin{cases} 1, & n = 0, \\ \prod_{0\le j\le n-1} \left(1 - aq^j\right), & n = 1,2,\ldots, \end{cases} \qquad (3)$$

where

$$\begin{aligned}(q^{-n};q)_k &= 0, \quad \text{whenever } n<k,\\ (0;q)_n &= 1.\end{aligned} \qquad (4)$$

The formula (3) is known as the Watson notation [4, 5]. The q-binomial coefficient [7, 10] is defined by

,

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k\,(q;q)_{n-k}}, \quad k,n\in\mathbb{N},$$

and for complex z is defined by

$$\begin{bmatrix} z \\ k \end{bmatrix}_q = \frac{(q^{-z};q)_k}{(q;q)_k}(-1)^k\, q^{zk-\binom{k}{2}}, \quad k\in\mathbb{N}. \qquad (5)$$

In addition, using the above definitions, we have that the binomial theorem

$$(x+y)^n = \sum_{0\le k\le n}\binom{n}{k}x^k b^{n-k}, \quad n=0,1,2,\ldots,$$

has a q-analogue of the form

$$\begin{aligned}(xy;q)_n &= \sum_{0\le k\le n}\begin{bmatrix} n \\ k \end{bmatrix}_q y^k\,(x;q)_k\,(y;q)_{n-k}\\ &= \sum_{0\le k\le n}\begin{bmatrix} n \\ k \end{bmatrix}_q x^{n-k}\,(x;q)_k\,(y;q)_{n-k}\end{aligned}$$

.

In particular, when $y = 0$ we have that

$$\sum_{0\le k\le n}\begin{bmatrix} n \\ k \end{bmatrix}_q x^{n-k}\,(x;q)_k = (0;q)_n = 1 \qquad (6)$$

In comparison with the ordinary hypergeometric series ${}_rF_s$ defined by (1), is present here in a concise manner, the basic hypergeometric or q-hypergeometric series ${}_r\varphi_s$. The details can be found in [7, 10].

Let $a_i\ {}_{i=0}^{r}$ and $b_j\ {}_{j=0}^{s}$ be complex numbers subject to the condition that $b_j \neq\ q^{-n}$ with $n\ \in\ \mathrm{N}\backslash\ 0$ for $j\ =\ 1,2,\ldots,s$. Then the basic hypergeometric or q-hypergeometric series ${}_r\varphi_s$ with variable z is defined by

$${}_r\varphi_s\left(\begin{array}{c} a_1,\ldots,a_r \\ \\ b_1,\ldots,b_s \end{array}\middle|\, q;z\right) \equiv \sum_{k>0}\frac{(a_1,\ldots,a_r;q)_k}{(b_1,\ldots,b_s;q)_k}(-1)^{(1+s-r)k}\,q^{(1+s-r)\binom{k}{2}}\frac{z^k}{(q;q)_k},$$

where

$$(a_1,\ldots,a_r;q)_k \equiv \prod_{1\le j\le r}(a_j;q)_k\,.$$

In addition, for brevity, let us denote by

$$\left[{}_r\varphi_s\left(\begin{array}{c}a_1,\ldots,a_r\\ \\ b_1,\ldots,b_s\end{array}\middle| q;z\right)\right]^n = {}_r\varphi_s^n\left(\begin{array}{c}a_1,\ldots,a_r\\ \\ b_1,\ldots,b_s\end{array}\middle| q;z\right), \quad n=1,2,\ldots.$$

The q-hypergeometric ${}_r\varphi_s$ series is a q-analogue of the ordinary hypergeometric ${}_rF_s$ series defined by (1) since

$$\lim_{q\to 1^-} {}_r\varphi_s\left(\begin{array}{c}q^{\tilde{a}_1},\ldots,q^{\tilde{a}_r}\\ \\ q^{\tilde{b}_1},\ldots,q^{\tilde{b}_s}\end{array}\middle| q;z\,(q-1)^{1+s-r}\right) = {}_rF_s\left(\begin{array}{c}\tilde{a}_1,\ldots,\tilde{a}_r\\ \\ \tilde{b}_1,\ldots,\tilde{b}_s\end{array}\middle| z\right).$$

The q-analogue of the Chu-Vandermonde convolution is given by

(7) $${}_2\varphi_1\left(\begin{array}{c}q^{-n},a\\ \\ b\end{array}\middle| q;\frac{bq^n}{a}\right) = \frac{(a^{-1}b;q)_n}{(b;q)_n}, \quad n=0,1,2,\ldots,$$

(8) $${}_2\varphi_1\left(\begin{array}{c}q^{-n},a\\ \\ b\end{array}\middle| q;q\right) = \frac{(a^{-1}b;q)_n}{(b;q)_n}a^n, \quad n=0,1,2,\ldots.$$

The details can be found in [7, 10].

The *q*-analogue $d_q f(x)$ of the differential of a fuction $df(x)$ is defined as

$$dq\, f(x) = f(qx) - f(x).$$

Having said this, we immediately get the *q*-analogue of the derivate of a function *f* (*x*) [6], called its *q*-derivative

$$\mathcal{D}_q f(x) = \frac{d_q f(x)}{d_q x} = \frac{f(qx)-f(x)}{(q-1)\,x}$$

The q-Jackson integrals [6] from 0 to a and from 0 to ∞ are defined by

$$\int_0^a f(x)\,d_q x = (1-q)\,a\sum_{n\geq 0} q^n f(aq^n),$$

$$\int_0^\infty f(x)\,d_q x = (1-q)\sum_{-\infty\leq n\leq\infty} q^n f(q^n),$$

provided the sums converge absolutely. The q-Jackson integral in a generic interval [a,b] is given by

$$\int_a^b f(x)\,d_qx = \int_0^b f(x)\,d_qx - \int_0^a f(x)\,d_qx.$$

A q-analogue of the integration by parts theorem is given for suitable functions f and g by

$$\int^b g(x)\,\mathcal{D}_q f(x)\,d_qx = f(x)\,g(x)\,\Big|_a^b - \int_a^b f(qx)\,\mathcal{D}_q g(x)\,d_qx,$$

and a q-analogue of the integration theorem by change of variable for $u(x) = \alpha x^{\beta}$, with $\alpha \in \mathbb{C}$ and $\beta > 0$ is as follows

$$\int_{u(b)}^{u(b)} f(u)\,d_qu = \int_a^b f(u(x))\,\mathcal{D}_{q^{1/\beta}}u(x)\,d_{q^{1/\beta}}$$

In particular

$$\begin{aligned}\int_0^1 z^{k-1}\log_q z d_qz &= \frac{1}{[k]_q}\lim_{\alpha\to 0} z^k \log_q z\Big|_\alpha^1 - \frac{q^k}{[k]_q}\int_0^1 z^k \mathcal{D}_q\left(\log_q z\right)d_qz \\ &= \frac{q^k(1-q)^{-1}}{[k]_q}\int_0^1 z^{k-1}d_qz = \frac{q^k(1-q)^{-1}}{[k]_q^2}.\end{aligned} \quad (9)$$

2 *q*-Analogue of the Apéry's constant

The Apéry's constant (2) has a *q*-analogue [8, 9, 13], defined by

$$\zeta_q(3) \equiv \sum_{n\geq 1}\frac{q^{2n}}{[n]_q^3} \quad (10)$$

where the *q*-integer $[n]_q$ is defined by

$$[n]_q \equiv \frac{1-q^n}{1-q} = \sum_{0\leq j\leq n-1} q^j$$

.

It makes sense to call this a *q*-analogue, since

$$\lim_{q\to 1^-}(1-q)^3\zeta_q(3) = 2!\zeta(3)$$

.

The *q*-analogue of the Apéry's constant $\zeta_q(3)$ is related with *q*-hypergeometric series ${}_4\varphi_3$ of the following way

$$\zeta_q(3) = q^2\sum_{n\geq 0}\frac{(q;q)_n^4}{(q^2;q)_n^3}\frac{q^{2n}}{(q;q)_n} = q^2{}_4\varphi_3\left(\begin{array}{c|c} q,q,q,q & \\ & q;q^2 \\ q^2,q^2,q^2 & \end{array}\right).$$

Lemma 3.1. The following relations

i.)

$${}_2\varphi_0\left(\begin{array}{c|c} z,q^{-n} & \\ & q;q^nz^{-1} \\ - & \end{array}\right) = z^{-n}, \quad n = 0,1,2,\ldots, \quad (11)$$

iii.)

$$\sum_{n\geq 1}\frac{1}{[n]_q^3} = {}_4\varphi_3\left(\begin{array}{c|c} q,q,q,q & \\ & q;1 \\ q^2,q^2,q^2 & \end{array}\right) = -\sum_{k\geq n\geq 1}\frac{q^{k-n}}{[k]_q^2[n]_q}.$$

holds.

(12)

Proof. Taking into account that

$$ {}_2\varphi_0\left(\begin{array}{c} z, q^{-n} \\ - \end{array}\middle| q; q^n z^{-1}\right) = \sum_{k\geq 0} \frac{(q^{-n};q)_k}{(q;q)_k} (-1)^k q^{nk-\binom{k}{2}} (z;q)_k z^{-k}. $$

Then, from (4) and (5) we have

$$ {}_2\varphi_0\left(\begin{array}{c} z, q^{-n} \\ - \end{array}\middle| q; q^n z^{-1}\right) = z^{-n} \sum_{0\leq k\leq n} \begin{bmatrix} n \\ k \end{bmatrix}_q z^{n-k} (z;q)_k . $$

Finally, using (6) we get the desired result for (11).

From the acquired result in (9) we get that

$$ \begin{aligned} \sum_{k\geq n\geq 1} \frac{q^{k-n}}{[k]_q^2 [n]_q} &= \sum_{n\geq 1} \frac{q^{-n}}{[n]_q} \sum_{k\geq n} \frac{q^k}{[k]_q^2} \\ &= (1-q) \sum_{n\geq 1} \frac{q^{-n}}{[n]_q} \sum_{k\geq n} \int_0^1 x^{k-1} \log_q x d_q x \\ &= (1-q) \sum_{n\geq 1} \frac{q^{-n}}{[n]_q} \int_0^1 \log_q x \sum_{k\geq n} x^{k-1} d_q x. \end{aligned} $$

The interchanges of summation and integration are in each case justified by Lebesgue’s monotone convergence theorem. Then

$$ \begin{aligned} \sum_{k\geq n\geq 1} \frac{q^{k-n}}{[k]_q^2 [n]_q} &= (1-q) \sum_{n\geq 1} \frac{q^{-n}}{[n]_q} \int_0^1 \log_q x \frac{x^n}{1-x} d_q x \\ &= (1-q) q^{-1} \int_0^1 \frac{\log_q x}{1-x} \sum_{n\geq 1} \frac{x^n}{[n]_q} q^{1-n} d_q x \\ &= (1-q) q^{-1} \int_0^1 \frac{\log_q x}{1-x} \log_q \left(\frac{1}{1-x}\right) d_q x. \end{aligned} $$

After making the change of variable t = 1 − x, we obtain

$$ \begin{aligned} \sum_{k\geq n\geq 1} \frac{q^{k-n}}{[k]_q^2 [n]_q} &= (q-1) q^{-1} \int_0^1 \frac{\log_q t}{t} \log_q \left(\frac{1}{1-t}\right) d_q t \\ &= (q-1) q^{-1} \int_0^1 \log_q t \sum_{n\geq 1} \frac{t^{n-1}}{[n]_q} q^{1-n} d_q t \\ &= (q-1) \sum_{n\geq 1} \frac{q^{-n}}{[n]_q} \int_0^1 t^{n-1} \log_q t d_q t. \end{aligned} $$

Therefore

$$\begin{aligned}\sum_{k\geq n\geq 1}\frac{q^{k-n}}{[k]_q^2\,[n]_q} &= -(1-q)\sum_{n\geq 1}\frac{q^{-n}}{[n]_q}\int_0^1 t^{n-1}\log_q t d_q t\\ &= -\sum_{n\geq 1}\frac{q^{-n}}{[n]_q}\frac{q^n}{[n]_q^2} = -\sum_{n\geq 1}\frac{1}{[n]_q^3}.\end{aligned}$$

which coincides with (12). In this way, the lemma is completed.

Theorem 3.2. Let $|q| < 1$, then the q-analogue of the Apéry's constant (10) admits the following representations

$$\zeta_q(3) = q(1-q)\int_0^1\int_0^1\frac{\log y}{1-qxy}d_qx d_qy, \qquad (13)$$

and

$$\begin{aligned}\zeta_q(3) = (q-1)\sum_{n\geq 1}{}_2\varphi_1^3&\left(\begin{array}{c} q^{-n}, q \\ \\ q^2\end{array}\middle|\, q; q^{n+1}\right)\\ &\times\sum_{1\leq j\leq 2n}{}_2\varphi_0\left(\begin{array}{c} q^{-1}, q^{-(j-1)} \\ \\ -\end{array}\middle|\, q; q^{j}\right)\\ &\qquad -\sum_{k\geq n\geq 1}\frac{q^{k-n}}{[k]_q^2\,[n]_q}. \qquad (14)\end{aligned}$$

Proof. In order to prove (13) it's enough check

$$\begin{aligned}q(1-q)\int_0^1\int_0^1\frac{\log y}{1-qxy}d_qxd_qy &= q(1-q)\int_0^1\int_0^1\sum_{n\geq 0}(qx)^n y^n\log y d_qx d_qy\\ &= q(1-q)\sum_{n\geq 0}q^n\int_0^1 x^n\left(\int_0^1 y^n\log y d_qx\right)d_qx\\ &= \sum_{n\geq 0}\frac{q^{2(n+1)}}{[n+1]_q^3} = \sum_{n\geq 1}\frac{q^{2n}}{[n]_q^3}.\end{aligned}$$

Then, having into account

$$\begin{aligned}\sum_{n\geq 1}\frac{q^{2n}-1}{[n]_q^3} &= (q-1)\sum_{n\geq 1}\frac{(q;q)_n^3}{(q^2;q)_n^3}\sum_{1\leq j\leq 2n}\left(q^{-1}\right)^{-(j-1)}\\ &= (q-1)\sum_{n\geq 1}{}_2\varphi_1^3\left(\begin{array}{c} q^{-n}, q \\ \\ q^2\end{array}\middle|\, q; q^{n+1}\right)\\ &\qquad\times\sum_{1\leq j\leq 2n}{}_2\varphi_0\left(\begin{array}{c} q^{-1}, q^{-(j-1)} \\ \\ -\end{array}\middle|\, q; q^{j}\right),\end{aligned}$$

and the q-Chu-Vandermonde formula (8) as well as the lemma 3.1, we get the desired result for (14). Thus, the prove is completed.

BIBLIOGRAPHIC REFERENCES

Alzer, H. (2003). Remark on a double-inequality for the Riemann zeta function, *Expo. Math*., Vol. 23, pp. 349–352.

Arvesú, J. (2012), Orthogonal forms: A key tool for deducing Apéry's recurrence relation, J. Approx. Theory, accepted.

Choi, J. and Srivastava, H. M. (2000). Certain classes of series associated with the Zeta function and multiple gamma functions, J. of Computational and Applied Mathematics, Vol. 118, pp. 87–109.

Ernst, T. (2003). A Method for q-Calculus, J. of *Nonlinear Mathematical Physics*, Vol. 10, No. 4, pp. 487–525.

Ernst, T. (2009). q-Calculus as operational algebra, *Proceedings of the Estonian Academy of Sciences*, Vol. 58, No. 2, pp. 73–97.

Fitouhi, A., Bettaibi, N., Binous, W., Elmonser, H. B., (2009). An uncertainty principle for the basic Bessel transform, Ramanujan J, DOI 10.1007/s11139-007-9117-6., Vol. 18, pp. 171–182.

Gasper, G. and Rahman. M. (2004). Basic hypergeometric series, Cambridge University Press.

Kaneko, M., Kurokawa, N. and Wakayama, M. A variation of Euler's approach to values of the Riemann zeta function, arXiv:math/0206171v2 [math.QA], (31 July 2012).

Kim, T. (2004). q-Riemann zeta function, IJMMS, PII. S0161171204307180, pp. 599–605.

Koekoek, R. and Swarttouw, R. F. (1998). The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue, Report 98-17, Faculty of Technical Mathematics and Informatics, Delft University of Technology.

Nesterenko, Yu. V. (2003). Integral identities and constructions of approximations to zeta values, J. de *Theorie des Nombres de Bordeaux*, Vol. 15, pp. 535–550.

Nikiforov, A. F., Uvarov, V. B. (1988). Special Functions in Mathematical Physics, Birkhauser Verlag, Basel.

Postelmans, K. and Van Assche, W. (2007). Irrationality of ζ q (1) and ζ q (2), J. Number Theory, Vol. 126, pp. 119–154.

Van Assche, W. (1999). Multiple orthogonal polynomials, irrationality and transcendence. *Contemporary Mathematics*, Vol. 236, pp. 325–342.

udilin, W., (2002). An elementary proof of Apéry's theorem, E-print math. NT/0202159, Moscow Lomonosov State University, pp. 1–8.